Lecture 3

Equivalence Relations

Since relations are defined as sets operations, this topic is directly connected to sets, also.

Definition. A relation $R \subset A^2$ is an equivalence relation if it is

1) reflective,
2) symmetric,
3) transitive.

Example 1. The identity relation I_A is an equivalence relation in set A.

<u>Proof</u>. a) R is reflective iff

$\forall a \in A \Rightarrow (a,a) \in R$. Definition:

$$\boxed{I_A = \{(a,a) : a \in A\}}$$

$\Rightarrow I_A$ is reflective.

b) R is symmetric iff $\forall a, b \in A$ such that

$(a,b) \in R \Rightarrow (b,a) \in R$.

$(a,b) \in I_A \Leftrightarrow b = a$. thus

$(b,a) = (a,a) \in R$.

c) R is transitive iff $\forall a, b, c \in A$

$(a,b) \in R$ and $(b,c) \in R \Rightarrow (a,c) \in R$.

$(a,b) \in I_A \Rightarrow a = b$

$\Rightarrow a = c$

$(b,c) \in I_A \Rightarrow b = c$

and $(a,c) = (a,a) \in I_A$.

Example 2.

$R_2 = A^2$, $A = \{1, 2, 3\}$

$R_2 = \{(1,1), (1,2), (2,1), (2,2), (3,3)\}$.

a) $(1,1) \in R_2$, $(2,2) \in R_2$, $(3,3) \in R_2$

$\Rightarrow R_2$ is reflective.

b) $(1,2) \in R_2$ and $(2,1) \in R_2$

R_2 is symmetric

c) $(1,2) \in R_2$, $(2,1) \in R_2$

and $(1,1) \in R_2$

R_2 is transitive.

Equivalence Classes

Let $R \subset A^2$ is an equivalence relation.

Definition. Subset $[a]_R \subset A$

$$[a]_R = \{ b \in A : (a, b) \in R \}$$

is called an equivalence class of element $a \in A$.

Lemma. The following statements are valid:

1.) $\forall a \in A : [a]_R \neq \emptyset$.

Proof. $a \in [a]_R$.

2) $\forall (a, b) \in R \Rightarrow [a]_R = [b]_R$.

Proof. $b \in [a]_R \Rightarrow (a, b) \in R$

$\overset{\text{symmetry}}{\Rightarrow}$ $(b, a) \in R \Rightarrow a \in [b]_R$.

Let's consider $c \in [a]_R$.

$\Rightarrow (a, c) \in R \overset{\text{transit}}{\Longrightarrow} (b, c) \in R$

Since $(b, a) \in R \Rightarrow$

$\Rightarrow c \in [b]_R$.

$\Rightarrow [a]_R \subseteq [b]_R$.

In a similar way it can be proved that

$[b]_R \subseteq [a]_R \Rightarrow [a]_R = [b]_R$.

3. $\forall (a, b) \notin R \Rightarrow [a]_R \cap [b]_R = \emptyset$.

<u>Proof.</u> Suppose $c \in [a]_R$, $c \in [b]_R$

$\Rightarrow (a, c) \in R, \ (b, c) \in R$

$\overset{\text{symmetry}}{\Longrightarrow} (c, b) \in R \overset{\text{transit}}{\Longrightarrow} (a, b) \in R$

We got a contradiction to the assumption of Lemma p. 3.

<u>Definition</u>. The set of all equivalence classes of A with respect to relation R is called the <u>quotient set</u> of A by R and denoted A/R

$$\boxed{A/R = \{[a]_R\}_{a \in A}}$$

<u>Example</u>. We say that a is <u>congruent</u> to b modulo m, and write $a \equiv b \bmod m$, $a, b \in \mathbb{Z}$.

$$a \equiv b \bmod m$$

if m divides the b-a.

$$-2 \equiv 19 \bmod 21, \quad 9 \equiv 1 \bmod 2,$$

$$10 \equiv 0 \bmod 2.$$

Verify that congruence modulo m is an equivalence relation on the integers

1) reflexivity, 2) symmetry,

3) transitivity.

The equivalence class of $\underline{a}$, i.e.

$$[a]_R = a + m\mathbb{Z}$$

consists of all integers that are obtained from a by adding <u>integer multiples of m</u>

$$a + m\mathbb{Z} = \{ b : b \equiv a \bmod m \}.$$

Example

$$2 + 5\mathbb{Z} = \{ 2, 2 \pm 5, 2 \pm 10, 2 \pm 15, \dots \}.$$

The quotient set of residue classes mod m is denoted $\mathbb{Z}/m\mathbb{Z}$. It has m elements, since $0, 1, 2, \dots, m-1$ are the possible <u>remainders</u> of the division by m.

$$\mathbb{Z}/5\mathbb{Z} = \{ 5\mathbb{Z}, 1+5\mathbb{Z}, 2+5\mathbb{Z}, 3+5\mathbb{Z}, 4+5\mathbb{Z} \}.$$

Order Relations

In our day ~~acti~~ jobs in general, and in mathematics, in particular, we compare various elements, sort them in some order:

- numbers can be sorted (compared)

  $3 < 7, \quad -1 \leq 8, \quad 10 > 4.$

- lexicographic order

  in dictionary 'apple' comes <u>before</u> 'orange' (or 'orange' comes <u>after</u> 'apple').

Now we define a general property of <u>order relations.</u>

~~Let's~~ Let's start from definition of <u>antisymmetric relations.</u>

<u>Definition</u>. A relation R in a set A is called an antisymmetric relation if from $(a, b) \in R$ and $(b, a) \in R$ it follows that $a = b$.

$$(a, b) \in R \;\&\; (b, a) \in R \Rightarrow a = b.$$

<u>Note</u> If only one of element (a, b) or (b, a) belongs to R then the condition of the definition is not needed. Thus <u>different</u> vertexes a and b can not be connected by both <u>directed edges</u> (a, b) and (b, a)

<u>Lemma</u>. R is antisymmetric <u>iff</u>

$$R \cap R^{-1} \subset I_A.$$

(Prove this statement)

<u>Definition</u>. A relation R on A is irreflexive if

$$\forall a \in A \quad (a,a) \notin R.$$

Note that <u>irreflexivity</u> is different from non-reflexivity.

<u>Every</u> irreflexive relation R on a non-empty set A is also <u>non-reflexive</u>, but a non-reflexive relation need not be irreflexive.

$$A = \{1, 2, 3\}$$

$$R_1 = \{(1,1), (1,2), (2,3)\}$$

$$R_2 = \{(1,1), (1,2), (2,2), (3,3), (2,3)\}$$

$$R_3 = \{(1,2), (1,3), (2,3)\}.$$

Definition. Binary relation R on set A is order relation if it is antisymmetric and transitive relation.

If additionally R is ~~antir~~ irreflexive (reflexive) then it is called a strict (weak) order relation.

A relation R is a total (partial) order relation if it is (is not) a complete relation.

Def. R is a complete relation (strongly connected) on A if

$$\forall a, b \in A \ \& \ a \neq b \Rightarrow$$
$$(a, b) \in R \vee (b, a) \in R.$$

Example.

Relation $R = \{(1,2), (1,3), (2,3)\}$

is complete on $A = \{1, 2, 3\}$.

But R is not complete

on $A = \{1, 2, 3, 4\}$.

It is clear, that if R is not complete then any $T \subset R$ is also not complete relation (give a proof!).

Examples.

Let's consider sets of numbers

$\mathbb{N}, \mathbb{Z}, \mathbb{R}$

Relations a) $\leq, \geq$

b) $<, >$

Proof. a1) $\leq$ is a weak order relation

Order relation : antisymmetric and transitive

Weak : reflective.

$(a, b) \in R : a \leq b \qquad a, b \in \mathbb{Z}$ (or $\mathbb{N}$, or $\mathbb{R}$)

antisymmetric.

If $(a, b) \in R$ then $(b, a) \notin R$

$a \neq b$

$a \leq b \overset{a \neq b}{\Rightarrow} a < b \Rightarrow b \nleq a \Rightarrow (b, a) \notin R.$

transitive

$a, b, c \in A \quad (A = \mathbb{N}, \mathbb{Z} \text{ or } \mathbb{R})$.

$(a, b) \in R$ and $(b, c) \in R$

$(a, c) \in R$?

$a \leq b \quad \& \quad b \leq c \Rightarrow a \leq c$

$\Rightarrow (a, c) \in R$

reflective

$\forall a \in A \Rightarrow a \leq a \Rightarrow (a, a) \in R$.

Thus $R =$ '$\leq$' is a weak order relation.

$R =$ '$>$' is a strong order relation.

It is sufficient to prove that R is a irreflective relation.

irreflective

$\forall a \in A \ (a, a) \notin R.$

$a R a \ = \ a > a$ is not valid

$(a, a) \notin R.$

Determine the type of relation

$A = \{1, 2, 3, 4, 5\}$

$R = \{(1,2), (1,5), (1,4), (2,5), (3,2), (3,5), (4,2)\}$

Lets consider a set of subsets of A

$A = \{a, b, c\}$

$2^A = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, A\}.$

Determine the type of relation '$\subset$'.

see 137 pd.

Ordered Sets

<u>Definition</u> A set is <u>totally</u> (<u>partially</u>) <u>ordered</u> if some relation of a total (partial) order R is defined on this set.

<u>Notation</u>. Ordered set $(A, \preceq)$ where R is denoted by $\preceq$

<u>Definition</u>. $m \in A$ is said to be a <u>minimal element</u> of the ordered set $(A, \preceq)$ if

$$\nexists a \in A : a \preceq m \ \& \ a \neq m .$$

For a partially ordered set $(A, \preceq)$ a minimal element <u>can be not unique</u>.

Theorem.

Any nonempty finite partially ordered set has its minimal element.

Any nonempty finite totally ordered set has the unique minimal element.

Example 1. Sets of numbers $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ are totally ordered by relations $\leq$ or $<$. Explain why they don't have minimal elements.

Example 2. The power set $P(A) = 2^A$ (the set of all subsets of set A) is a partially ordered set if the order is introduced by relation $\subset$. Find a minimal element in P.

Example 3. Let's take a set

$$A = \{2, 3, 4, 6\} \subset \mathbb{N}$$

and consider the relation $R \subset A^2$

$$R = \{(x, y) : y = kx \text{ for some } k \in \mathbb{N}\}.$$

1. Find all elements of R.

$$R = \{(2,2), (2,4), (2,6), (3,3), (3,6), (4,4), (6,6)\}.$$

2. Determine the type of this relation.
3. Find minimal element / elements.

p. 138.